Existence and uniqueness for a coupled parabolic-elliptic model with applications to magnetic relaxation

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Abstract We prove existence, uniqueness and regularity of weak solutions of a coupled parabolic-elliptic model in 2D, and existence of weak solutions in 3D; we consider the standard equations of magnetohydrodynamics with the advective terms removed from the velocity equation. Despite the apparent simplicity of the model, the proof in 2D requires results that are at the limit of what is available, including elliptic regularity in L^1 and a strengthened form of the Ladyzhenskaya inequality

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|_{L^2}^{1/2},$$

which we derive using the theory of interpolation. The model potentially has applications to the method of magnetic relaxation introduced by Moffatt (J. Fluid. Mech. **159**, 359–378, 1985) to construct stationary Euler flows with non-trivial topology.

1 Introduction

In this paper we prove global existence and uniqueness of solutions to the following coupled parabolic-elliptic system of equations related to magnetohydrodynamics (MHD), for a velocity field $\boldsymbol{u}$, a magnetic field $\boldsymbol{B}$ and a

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pressure field p, defined on a domain Ω in two or three dimensions, as follows:

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.1a}$$
$$\partial_t\boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.1b}$$
$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0. \tag{1.1c}$$

Here $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure, $\nu > 0$ is the coefficient of viscosity, and $\eta > 0$ is the coefficient of magnetic resistivity. The domain Ω may be one of three cases:

- $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain;
- $\Omega = \mathbb{R}^n$; or
- $\Omega = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

In the first case, we consider (1.1) with Dirichlet boundary conditions; in the third case we consider periodic boundary conditions on $[0,1]^n$.

This model has interesting analogies with the vorticity formulation of the 3D Navier–Stokes and Euler equations, as well as with the 2D surface quasi-geostrophic equations. Recall that the vorticity formulation of the Navier–Stokes equations in three dimensions is

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - \eta\Delta\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}, \tag{1.2}$$

where $\boldsymbol{u} = K * \boldsymbol{\omega}$ is given by the Biot–Savart law, with K a homogeneous kernel of degree $1-n$ in dimension n. Our two-dimensional model has a very similar form — compare (1.2) with (1.1b) — but $\boldsymbol{u}$ is instead given by

$$\boldsymbol{u} = K * (\boldsymbol{B}\otimes\boldsymbol{B}),$$

where K involves derivatives of the fundamental solution of the Stokes equation, and is homogeneous of degree $1-n$. Unlike the 3D Navier–Stokes equations, for which existence and uniqueness of solutions for all time remains open, our two-dimensional model retains the essential features of the nonlinearities but admits a unique solution for all time.

Unusually, the existence proof for the two-dimensional case is harder than the three-dimensional case, so we focus on 2D in this paper. Indeed, the main purpose of this paper is to prove the following theorem on the existence and uniqueness of weak solutions of (1.1).

First, let us define $\mathcal{D}_\sigma(\Omega) := \{\boldsymbol{u}\in C_c^\infty(\Omega) : \nabla\cdot\boldsymbol{u} = 0\}$ in the case when Ω is a bounded domain in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, and let $\mathcal{D}_\sigma(\mathbb{T}^n) := \{\boldsymbol{u}\in C^\infty(\mathbb{T}^n) : \nabla\cdot\boldsymbol{u} = 0\}$ (with the understanding that such $\boldsymbol{u}$ are periodic). Let $V(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the H^1 norm, let $H(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the L^2 norm, and finally let $V^*(\Omega)$ denote the dual of $V(\Omega)$.

Theorem 1.1 *Let Ω be one of the following:*

- *$\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain;*
- *$\Omega = \mathbb{R}^2$; or*
- *$\Omega = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.*

Given an initial condition $\boldsymbol{B}_0 \in H(\Omega)$*, for any* $T > 0$ *there exists a unique pair of functions* $(\boldsymbol{u}, \boldsymbol{B})$ *with*

$$\begin{aligned}\boldsymbol{u} &\in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \boldsymbol{B} &\in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \partial_t \boldsymbol{B} &\in L^2(0,T;V^*(\Omega)),\end{aligned}$$

such that $\boldsymbol{B}(0) = \boldsymbol{B}_0$ *and for almost every* $t \in (0,T)$

$$\begin{aligned}0 &= \nu\langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v}_1\rangle + \langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_1, \boldsymbol{B}\rangle,\\ \langle \partial_t \boldsymbol{B}, \boldsymbol{v}_2\rangle &= \eta\langle \nabla \boldsymbol{B}, \nabla \boldsymbol{v}_2\rangle + \langle (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_2, \boldsymbol{u}\rangle - \langle (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_2, \boldsymbol{B}\rangle,\end{aligned}$$

for every pair of functions $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V(\Omega)$*. Furthermore, for any* $T > \varepsilon > 0$ *and any* $k \in \mathbb{N}$*,*

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(\varepsilon,T;H^k(\Omega)) \cap L^2(\varepsilon,T;H^{k+1}(\Omega)).$$

We also prove the existence of at least one weak solution to (1.1) in 3D; see Section 6. Note, however, that we do not prove that such weak solutions are unique, unlike the 2D case above.

Our interest in system (1.1) arises from its connection with the method of *magnetic relaxation*, as discussed by Moffatt [32]. He considers the related full MHD system:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{1.3a}$$

$$\partial_t \boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{1.3b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0. \tag{1.3c}$$

Formally, when $\eta = 0$, we obtain the standard energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2\right) + \nu\|\nabla\boldsymbol{u}\|_{L^2}^2 = 0;$$

thus while as $\boldsymbol{u}$ is not identically zero, the energy should decay. Furthermore, by using the so-called *magnetic helicity*, which is preserved under the flow, we can find a lower bound for the energy of $\boldsymbol{B}$: if $\mathscr{H}_M := \int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}$, where $\nabla\times\boldsymbol{A} = \boldsymbol{B}$ is a vector potential for $\boldsymbol{B}$, then

$$C\|\boldsymbol{B}\|_{L^2}^4 \geq \|\boldsymbol{B}\|_{L^2}^2\|\boldsymbol{A}\|_{L^2}^2 \geq \left(\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\right)^2 = |\mathscr{H}_M|^2 > 0.$$

In other words, the magnetic forces on a viscous non-resistive plasma should come to equilibrium, so that the fluid velocity $\boldsymbol{u}$ tends to zero. We are left with a steady magnetic field $\boldsymbol{B}$ that satisfies $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla p_* = 0$, which up to a change of sign for the pressure are the stationary Euler equations.

These arguments are heuristic, and as yet there is no rigorous proof that this method will yield a stationary Euler flow. The first problem is that it has not yet been proved that the system (1.3) with $\eta = 0$ has a unique solution for all time, even in two dimensions: short-time existence of strong solutions is proved by means of a "vanishing resistivity" argument in [22], while in [17] we reduce the regularity required to show short-time existence. A conditional regularity result was also proved in [22], and was later extended in [16].

With $\eta > 0$, however, the existence theory for (1.3) is in a similar state to the Navier–Stokes equations, with global existence of weak solutions in two or three dimensions, and uniqueness in two dimensions; see [13] and [36]. (Interestingly, global existence of weak solutions in two dimensions for the case $\nu = 0$ but $\eta > 0$ was proven in [24], with various extensions in [6] and [7], and conditional regularity results in [16] and [39].)

The second problem is that, even with global existence and uniqueness, the system may not possess a limit state. Assuming that the equations have a smooth solution for all time, and furthermore that $\|\boldsymbol{B}\|_\infty \le M$ for all time, Núñez [35] showed that (with $\eta = 0$) the kinetic energy must decay to zero, but that the magnetic field may not have a weak limit when a decaying forcing $f \in L^2(0,\infty;L^2(\Omega))$ is added to the $\boldsymbol{u}$ equation.

On the other hand, one can examine whether or not a stationary Euler flow with a given topology exists, without reference to any dynamical model. Indeed, the existence of a stationary Euler flow, albeit with infinite energy, with stream or vortex lines of prescribed link type was proved in [14]; but whether such flows arise as limits of system (1.3) is still very much open.

However, since the dynamical model used to obtain that steady state is not particularly important, it might prove fruitful to consider an alternative model for magnetic relaxation. In a talk given at the University of Warwick, Moffatt [33] argued that dropping the acceleration terms from the $\boldsymbol{u}$ equation and working with a "Stokes" model — such as equations (1.1) — might prove more mathematically amenable. As a first step towards a rigorous theory of magnetic relaxation for this model, this paper thus establishes existence and uniqueness theory for (1.1) in the case $\eta > 0$ in two dimensions, and existence theory in three dimensions.

The proof of Theorem 1.1 is divided into several sections:

- In Section 2, we use the theory of interpolation spaces to prove the following generalised version of the 2D Ladyzhenskaya inequality:

$$\|f\|_{L^{p,r}} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}$$

  for $1 < q < p < \infty$ and $1 \le r \le \infty$ (in particular, we are interested in the case $p = r = 4$, $q = 2$).
- In Section 3 we consider elliptic regularity for the Stokes equations

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \nabla\cdot\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned}$$

  in 2D, and show that $\boldsymbol{u} \in L^{2,\infty}$ whenever $\boldsymbol{f} \in L^1$.
- In Section 4, we use the results of the previous two sections to prove global existence and uniqueness of (L^2-valued) weak solutions to (1.1) in a bounded domain Ω and the whole of $\mathbb{R}^2$.
- In Section 5 we prove higher-order estimates to show that the solutions stay as smooth as the initial data permits for all time, and hence that after any arbitrary time $\varepsilon > 0$ the solution is smooth.

Finally, in Section 6, we outline the changes necessary to prove existence (but not uniqueness) of weak solutions to (1.1) in three dimensions, both in the case where $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the whole space $\Omega = \mathbb{R}^3$.

Before we begin our formal treatment of the problem, it is instructive to note that (in the whole space case) the equations (1.1) are invariant under the rescaling

$$\boldsymbol{u}(x,t) \mapsto \lambda \boldsymbol{u}(\lambda x, \lambda^2 t), \qquad \boldsymbol{B}(x,t) \mapsto \lambda \boldsymbol{B}(\lambda x, \lambda^2 t).$$

In the two-dimensional case (which is the main focus of this paper) the critical (scale-invariant) spaces include the natural energy space L^2 in which we pose the problem for $\boldsymbol{B}$, and the space $L^{2,\infty}$ in which the corresponding velocity field $\boldsymbol{u}$ then lies (due to the elliptic nature of equation (1.1a), see Section 3).

Existence results for the three-dimensional Navier–Stokes equations in such critical spaces have received much attention in recent years (see [28] for an extensive summary; the result of Koch & Tataru [23] in BMO^{-1} is generally considered definitive), the standard technique being to recast the equations in integral form and seek the solution as the fixed point of the resulting integral operator in an appropriately chosen Banach space. However, for L^2-valued weak solutions (in 2D and 3D) of the kind we study here, it is more usual (and significantly simpler) to employ a proof based on the Galerkin method and relatively elementary energy estimates.

2 Interpolation and Ladyzhenskaya's inequality

In order to prove existence and uniqueness for our system (1.1), we will require a variant of Ladyzhenskaya's inequality. We first recall the standard inequality proved by Ladyzhenskaya [27]: if $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, then for $u \in H^1(\Omega)$,

$$\|u\|_{L^4} \le c\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}. \tag{2.1}$$

(One can prove this simply by using the embedding $H^{1/2} \subset L^4$ and interpolating $H^{1/2}$ between L^2 and H^1.)

The variant of Ladyzhenskaya's inequality that we require is the standard inequality with $\|u\|_{L^2}$ replaced with $\|u\|_{L^{2,\infty}}$, where $L^{2,\infty}$ denotes the weak L^2 space:

$$\|f\|_{L^4} \le c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|_{L^2}^{1/2}. \tag{2.2}$$

In fact we will use the theory of interpolation spaces to prove the following stronger inequality involving the Lorentz space $L^{p,q}(\mathbb{R}^n)$ (see, e.g., [20], §1.4):

$$\|f\|_{L^{p,r}} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p} \tag{2.3}$$

for every $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, when $1 < q < p < \infty$, and $1 \le r \le \infty$.

The inequality (2.3) is not altogether new: an alternative proof is sketched in [25]. Furthermore, it is a strengthening of the inequality

$$\|f\|_{L^p} \le c\|f\|_{L^q}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}, \tag{2.4}$$

which has been proved a number of times before; see [9], [26], [12] and [2]. In particular, the elegant proof of (2.4) in Chen & Zhu [9], which uses the John–Nirenberg inequality for functions in BMO, is adapted in McCormick et al. [30] to give a proof of (2.3) in the case $p = r$ (i.e. with just the L^p norm on the left-hand side) that bypasses the use of interpolation spaces.

A related interpolation inequality involving Besov spaces is proved in [3], Theorem 2.42:

$$\|f\|_{L^p} \le c\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{1-2/p}\|f\|_{\dot{H}^1}^{2/p} \qquad \text{for } \alpha = \frac{1}{p/2-1}. \tag{2.5}$$

Here we prove our weak version of Ladyzhenskaya's inequality, using some of the standard theory of interpolation spaces. We recall here only the basic facts we require: for full details, see the books of Bennett and Sharpley [4], §5.1, and Bergh and Löfström [5], §3.1.

Given two compatible Banach spaces X_0, X_1 (that is, there is a Hausdorff topological vector space $\mathfrak{X}$ such that X_0 and X_1 embed continuously into $\mathfrak{X}$), and θ and q such that either $0 < \theta < 1$ and $1 \le q < \infty$, or $0 \le \theta \le 1$ and $q = \infty$, there exists an interpolation space $(X_0, X_1)_{\theta,q}$, such that

$$\|f\|_{\theta,q} \le c\|f\|_{X_0}^{1-\theta}\|f\|_{X_1}^{\theta} \tag{2.6}$$

(see [5], §3.5, p. 49). As a simple example of interpolation, note that

$$(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1-1/p,q} = L^{p,q}(\mathbb{R}^n)$$

if $1 < p < \infty$ and $1 \le q \le \infty$ (see [4], Chapter 5, Theorem 1.9). In fact, this equality remains true with L^∞ replaced with BMO (see [4], Chapter 5, Theorem 8.11):

$$(L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1-1/p,q} = L^{p,q}(\mathbb{R}^n). \tag{2.7}$$

This will be key for our equations, since $\dot{H}^{n/2} \subset \mathrm{BMO}$ but $\dot{H}^{n/2} \not\subset L^\infty$.

The so-called *reiteration theorem* says that when we interpolate between two interpolation spaces of the same couple (X_0, X_1), we get another interpolation space in the same family.

Theorem 2.1 (Reiteration Theorem) *Let (X_0, X_1) be compatible Banach spaces, let $0 \le \theta_0 < \theta_1 \le 1$, and let $1 \le q_0, q_1 \le \infty$. Set $A_0 = (X_0, X_1)_{\theta_0,q_0}$ and $A_1 = (X_0, X_1)_{\theta_1,q_1}$. If $0 < \theta < 1$ and $1 \le q \le \infty$, then*

$$(A_0, A_1)_{\theta,q} = (X_0, X_1)_{\theta',q}$$

providing $\theta' = (1-\theta)\theta_0 + \theta\theta_1$.

The proof may be found in [4], Chapter 5, Theorem 2.4, or [5], Theorem 3.5.3. Using this, we can prove our generalised Ladyzhenskaya inequality (2.3).

Lemma 2.2 (Interpolation) *Let $1 < q < p < \infty$, and $1 \le r \le \infty$. For any $f \in L^{q,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$,*

$$\|f\|_{L^{p,r}} \le c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}.$$

Proof Using (2.7), we have

$$L^{q,\infty}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1-1/q,\infty}$$

provided that $1 < q < \infty$. Set $\mathfrak{B} := (L^1(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))_{1,\infty}$, and note that by (2.6) we have $\|f\|_{\mathfrak{B}} \leq C\|f\|_{\mathrm{BMO}}$. By the Reiteration Theorem (Theorem 2.1), we obtain

$$L^{p,r}(\mathbb{R}^n) = (L^{q,\infty}(\mathbb{R}^n), \mathfrak{B})_{\alpha,r}$$

with $q < p < \infty$, provided that $\alpha = 1 - q/p$. Thus, using (2.6), we obtain

$$\|f\|_{L^{p,r}} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathfrak{B}}^{1-q/p} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p},$$

as required. □

One can prove that $\dot{H}^{n/2} \subset \mathrm{BMO}$ in n dimensions: see Theorem 1.48 in [3]. In particular, in two dimensions $\dot{H}^1 \subset \mathrm{BMO}$; so, for $f \in L^{2,\infty}(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$, setting $n = 2$, $p = r = 4$ and $q = 2$ in Lemma 2.2 we obtain (2.2):

$$\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{1/2}\|\nabla f\|_{L^2}^{1/2}.$$

When Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, we may extend a function $f \in H_0^1(\Omega)$ by zero outside Ω and apply the above inequality on $\mathbb{R}^2$ to obtain the same for such f.

When $\Omega = \mathbb{T}^2$, however, a different argument using Fourier series is required: we obtain the same inequality using the Sobolev embedding $L^4 \subset \dot{H}^{1/2}$ and the fact that

$$f = \sum_{|k|\leq\kappa} \hat{f}_k \mathrm{e}^{2\pi\mathrm{i}k\cdot x} \qquad \Longrightarrow \qquad \|f\|_{L^4} \leq c\kappa^{1/2}\|f\|_{L^{2,\infty}} \tag{2.8}$$

(a weak form of Bernstein's inequality; see [34] for the relevant theory of Fourier series and McCormick et al. [30] for the proof). Indeed, writing

$$f = \sum_{|k|\leq\kappa} \hat{f}_k \mathrm{e}^{2\pi\mathrm{i}k\cdot x} + \sum_{|k|>\kappa} \hat{f}_k \mathrm{e}^{2\pi\mathrm{i}k\cdot x}$$

we obtain, using (2.8) and $L^4 \subset \dot{H}^{1/2}$,

$$\begin{aligned}
\|f\|_{L^4} &\leq c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\left(\sum_{|k|>\kappa} |k||f_k|^2\right)^{1/2} \\
&\leq c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\kappa^{-1/2}\left(\sum_{|k|>\kappa} |k|^2|f_k|^2\right)^{1/2} \\
&\leq c\kappa^{1/2}\|f\|_{L^{2,\infty}} + c\kappa^{-1/2}\|\nabla f\|_{L^2}.
\end{aligned}$$

Minimising over κ we obtain (2.2). A similar argument involving Fourier transforms can be used to obtain a more general version of (2.2) and (2.3) (in the case $p = r$) on the whole space; see McCormick et al. [30].

3 The Stokes operator and elliptic regularity in L^1

We now consider the Stokes equation

$$-\nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{3.1}$$

on one of the three domains in Theorem 1.1, with Dirichlet boundary conditions if Ω is bounded. By setting $\boldsymbol{f} = \boldsymbol{B}\otimes\boldsymbol{B}$ (i.e. $\boldsymbol{f}_{i,j} = \boldsymbol{B}_i\boldsymbol{B}_j$) we recover equation (1.1a), since $\boldsymbol{B}$ is divergence-free. In this case, if $\boldsymbol{B}\in L^2(\Omega)$, then $\boldsymbol{B}\otimes\boldsymbol{B}$ is in $L^1(\Omega)$, so the right-hand side behaves like the derivative of an L^1 function. If $\boldsymbol{f}\in L^p(\Omega)$ for $p>1$, one would expect that $\boldsymbol{u}\in W^{1,p}(\Omega)$, but this does not hold for $p=1$. If it did, in two dimensions we would obtain $\boldsymbol{u}\in W^{1,1}(\Omega)\subset L^2(\Omega)$. In fact, in this section we prove that, when $\boldsymbol{f}\in L^1(\Omega)$ in (3.1), then $\boldsymbol{u}\in L^{2,\infty}(\Omega)$.

The solution of equation (3.1) is given by integration against the Green's function: let $\boldsymbol{U}$, q solve

$$-\nu\Delta\boldsymbol{U}(x,y) + \nabla q(x,y) = \delta(x-y), \qquad \nabla\cdot\boldsymbol{U} = 0,$$

where δ denotes the Dirac delta function. Then the solution of (3.1) is given by

$$\boldsymbol{u} = \int_\Omega \boldsymbol{U}(x,y)(\nabla\cdot\boldsymbol{f}(y))\,\mathrm{d}y, \qquad p = \int_\Omega q(x,y)(\nabla\cdot\boldsymbol{f}(y))\,\mathrm{d}y.$$

Integrating by parts with respect to k, we obtain

$$\boldsymbol{u}_i(x) = -\sum_{j,k=1}^{2}\int_{\mathbb{R}^2}\partial_k\boldsymbol{U}_{i,j}(x,y)f_{k,j}(y)\,\mathrm{d}y.$$

In the case $\Omega=\mathbb{R}^2$, we have explicit formulae for $\boldsymbol{U}$ and q: with abuse of notation, $\boldsymbol{U}_{i,j}(x,y) = \boldsymbol{U}_{i,j}(x-y)$ and $q_{i,j}(x,y) = q_{i,j}(x-y)$, where

$$\boldsymbol{U}_{i,j}(x) = \frac{1}{4\pi\nu}\left[\frac{x_ix_j}{|x|^2} - \delta_{ij}\log|x|\right], \qquad q_j(x) = \frac{1}{2\pi}\frac{x_j}{|x|^2}$$

(see [19], §IV.2). Direct calculation yields $|\partial_k\boldsymbol{U}_{i,j}(x)| \le \frac{1}{\pi\nu|x|}\in L^{2,\infty}(\mathbb{R}^2)$, so by Young's inequality for convolutions we obtain

$$\|\boldsymbol{u}\|_{L^{2,\infty}} \le c\|\partial_k\boldsymbol{U}_{i,j}\|_{L^{2,\infty}}\|\boldsymbol{f}\|_{L^1} \le c\|\boldsymbol{f}\|_{L^1}. \tag{3.2}$$

Thus, whenever $\boldsymbol{f}\in L^1(\mathbb{R}^2)$, $\boldsymbol{u}\in L^{2,\infty}(\mathbb{R}^2)$.

In the case where $\Omega=\mathbb{T}^2$, one can also write down an explicit formula for the fundamental solution — see [21] and [10], for example — and obtain (3.2) again; the details are very similar to the above case, and we omit them.

In the case where Ω is a bounded Lipschitz domain, while we no longer have an explicit formula for the Green's function $\boldsymbol{U}$, by Theorem 7.1 in [31] we have $\nabla\boldsymbol{U}(x,\cdot)\in L^{2,\infty}(\Omega)$ uniformly for $x\in\Omega$. Using a straightforward generalisation of Young's inequality to expressions of the form $\int_\Omega G(x,y)f(y)\,\mathrm{d}y$ when G is symmetric, we obtain (3.2) on a bounded Lipschitz domain as well; i.e. whenever $\boldsymbol{f}\in L^1(\Omega)$, $\boldsymbol{u}\in L^{2,\infty}(\Omega)$.

4 Existence and uniqueness of weak solutions

We return now to the system

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{4.1a}$$
$$\partial_t\boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \tag{4.1b}$$
$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{4.1c}$$

where $p_* = p + \frac{1}{2}|\boldsymbol{B}|^2$. We will show that equations (4.1) have a unique weak solution for all time in the three cases of Ω described in Theorem 1.1.

First, let us recall from the introduction the spaces in which we will work. Let $\mathcal{D}_\sigma(\Omega) := \{\boldsymbol{u} \in C_c^\infty(\Omega) : \nabla\cdot\boldsymbol{u} = 0\}$ in the case when Ω is a bounded domain in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, and let $\mathcal{D}_\sigma(\mathbb{T}^n) := \{\boldsymbol{u} \in C^\infty(\mathbb{T}^n) : \nabla\cdot\boldsymbol{u} = 0\}$. Let $V(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the H^1 norm, let $H(\Omega)$ be the closure of $\mathcal{D}_\sigma(\Omega)$ in the L^2 norm, and finally let $V^*(\Omega)$ denote the dual of V.

We first define a weak solution, in line with the terminology commonly used for the Navier–Stokes equations (see, e.g., [38]).

Definition 4.1 A pair of functions $(\boldsymbol{u},\boldsymbol{B})$ is a *weak solution* of (4.1) on $(0,T)$ if

$$\begin{aligned}\boldsymbol{u} &\in L^\infty(0,T;L^{n/(n-1),\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \boldsymbol{B} &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \partial_t\boldsymbol{B} &\in L^1(0,T;V^*(\Omega)),\end{aligned}$$

such that for almost every $t \in (0,T)$

$$\begin{aligned}0 &= \nu\langle\nabla\boldsymbol{u},\nabla\boldsymbol{v}_1\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_1,\boldsymbol{B}\rangle,\\ \langle\partial_t\boldsymbol{B},\boldsymbol{v}_2\rangle &= \eta\langle\nabla\boldsymbol{B},\nabla\boldsymbol{v}_2\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}_2,\boldsymbol{u}\rangle - \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_2,\boldsymbol{B}\rangle,\end{aligned}$$

for every pair of functions $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V(\Omega)$.

Note that the pressure p is uniquely determined up to an additive function of time by $\boldsymbol{u}$ and $\boldsymbol{B}$ by solution of a standard elliptic boundary value problem; see the discussion around equation (4.3) and the books [8] and [18].

In this section we will prove the following theorem.

Theorem 4.2 *Given $\boldsymbol{B}_0 \in H(\Omega)$, for any $T > 0$ there exists a unique weak solution $(\boldsymbol{u},\boldsymbol{B})$ of (4.1) on $(0,T)$, such that $\partial_t\boldsymbol{B} \in L^2(0,T;V^*(\Omega))$ and $\boldsymbol{B} \in C^0([0,T];L^2(\Omega))$, with $\boldsymbol{B}(0) = \boldsymbol{B}_0$.*

In Section 4.1, we will prove existence of a weak solution in the case $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain with Dirichlet boundary conditions, while in Section 4.2 we prove existence of a weak solution in the case $\Omega = \mathbb{R}^2$. The proof of existence in the case where $\Omega = \mathbb{T}^2$ is similar to the previous two, and we omit it. Finally, in Section 4.3, we prove uniqueness of weak solutions.

4.1 Global existence of solutions in a bounded domain

In this subsection we prove existence of a weak solution on a Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$, with Dirichlet boundary conditions. As with the 2D Navier–Stokes equations, we use energy methods and Galerkin approximations. To do so, we first set up some notation.

Let Π be the Leray projection $\Pi \colon L^2(\Omega) \to H$, i.e. the orthogonal projection from L^2 onto H. We define the *Stokes operator* as $A := -\Pi\Delta$. Let $\{\phi_m\}_{m\in\mathbb{N}} \subset C^\infty(\Omega)$ be the collection of eigenfunctions of the Stokes operator on Ω with Dirichlet boundary conditions, ordered such that the eigenvalues associated to ϕ_m are non-decreasing with respect to m. Let V_m be the subspace of H spanned by $\phi_1, \ldots, \phi_m$, and let $P_m \colon H \to V_m$ be the orthogonal projection onto V_m.

In order to use the Galerkin method, we consider the equations

$$-\nu\Delta \boldsymbol{u}^m + \nabla p_*^m = (\boldsymbol{B}^m \cdot \nabla)\boldsymbol{B}^m, \tag{4.2a}$$

$$\partial_t \boldsymbol{B}^m + P_m[(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{B}^m] - \eta\Delta \boldsymbol{B}^m = P_m[(\boldsymbol{B}^m \cdot \nabla)\boldsymbol{u}^m], \tag{4.2b}$$

$$\nabla \cdot \boldsymbol{u}^m = \nabla \cdot \boldsymbol{B}^m = 0. \tag{4.2c}$$

Note that we do not require a P_m on the right-hand side of (4.2a): this will make some of our convergence arguments easier (see Proposition 4.4).

Thinking of $\boldsymbol{u}^m$ as a function of $\boldsymbol{B}^m$ given by equation (4.2a), it is easy to check that (4.2b) is a locally Lipschitz ODE on the finite-dimensional space V_m, and thus by existence and uniqueness theory for finite-dimensional ODEs (Picard's theorem), there exists a unique solution $\boldsymbol{B}^m \in V_m$ of equation (4.2b), with $\boldsymbol{u}^m$ given by equation (4.2a).

Proposition 4.3 (Energy estimates) *The Galerkin approximations are uniformly bounded in the following senses:*

$$\begin{aligned} \boldsymbol{u}^m &\ \textit{is uniformly bounded in } L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \boldsymbol{B}^m &\ \textit{is uniformly bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \partial_t\boldsymbol{B}^m &\ \textit{is uniformly bounded in } L^2(0,T;V^*(\Omega)). \end{aligned}$$

Proof Take the inner product of equation (4.2a) with $\boldsymbol{u}^m$ and the inner product of equation (4.2b) with $\boldsymbol{B}^m$, and add to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}^m(t)\|_{L^2}^2 + \nu\|\nabla\boldsymbol{u}^m(t)\|_{L^2}^2 + \eta\|\nabla\boldsymbol{B}^m(t)\|_{L^2}^2 = 0.$$

Integrating over $[0,t]$ we obtain

$$\begin{aligned} \|\boldsymbol{B}^m(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla\boldsymbol{u}^m(s)\|_{L^2}^2\,\mathrm{d}s + 2\eta\int_0^t \|\nabla\boldsymbol{B}^m(s)\|_{L^2}^2\,\mathrm{d}s&\\ = \|\boldsymbol{B}^m(0)\|_{L^2}^2 \le \|\boldsymbol{B}_0\|_{L^2}^2,& \end{aligned}$$

so $\boldsymbol{u}^m \in L^2(0,T;V(\Omega))$ and $\boldsymbol{B}^m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega))$.

As in Section 3, the solution $\boldsymbol{u}^m$ to equation (4.2a) is given by convolution with $\boldsymbol{U}$, the Green's function for the Stokes equations. By (3.2), we have

$$\|\boldsymbol{u}^m(t)\|_{L^{2,\infty}} \le c\|(\boldsymbol{B}^m(t))^2\|_{L^1} \le c\|\boldsymbol{B}^m(t)\|_{L^2}^2,$$

so $\boldsymbol{u}^m \in L^\infty(0,T;L^{2,\infty}(\Omega))$.

For the estimate on $\partial_t \boldsymbol{B}^m$, taking the norm in V^* of the $\boldsymbol{B}$ equation yields

$$\|\partial_t \boldsymbol{B}^m\|_{V^*} \le \eta\|\boldsymbol{B}^m\|_V + \|P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m]\|_{V^*} + \|P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\|_{V^*}.$$

For $\phi \in V(\Omega)$,

$$|\langle P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m],\phi\rangle| \le \|\boldsymbol{B}^m\|_{L^4}\|\boldsymbol{u}^m\|_{L^4}\|\nabla\phi\|_{L^2},$$

so $\|P_m[(\boldsymbol{B}^m\cdot\nabla)\boldsymbol{u}^m]\|_{V^*} \le \|\boldsymbol{B}^m\|_{L^4}\|\boldsymbol{u}^m\|_{L^4}$ (and the same for the other term). By applying Ladyzhenskaya's inequality (2.1) to $\boldsymbol{B}^m$ and our weak Ladyzhenskaya's inequality (2.2) to $\boldsymbol{u}^m$, we obtain the the following estimate:

$$\|\partial_t \boldsymbol{B}^m\|_{V^*}^2 \le \eta\|\boldsymbol{B}^m\|_V^2 + c\|\boldsymbol{B}^m\|_{L^2}\|\boldsymbol{B}^m\|_V\|\boldsymbol{u}^m\|_{L^{2,\infty}}\|\boldsymbol{u}^m\|_V,$$

as required. □

To extract a convergent subsequence of $\boldsymbol{B}^m$, we use the Banach–Alaoglu theorem to extract a subsequence, which we relabel as $\boldsymbol{B}^m$, such that

$$\begin{aligned}
\boldsymbol{B}^m &\overset{*}{\rightharpoonup} \boldsymbol{B} && \text{in } L^\infty(0,T;L^2(\Omega)),\\
\boldsymbol{B}^m &\rightharpoonup \boldsymbol{B} && \text{in } L^2(0,T;V(\Omega)),\\
\partial_t\boldsymbol{B}^m &\overset{*}{\rightharpoonup} \partial_t\boldsymbol{B} && \text{in } L^2(0,T;V^*(\Omega)).
\end{aligned}$$

Since the limit $\boldsymbol{B} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega))$, it is straightforward to show that $(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \in L^2(0,T;V^*(\Omega))$. This allows us to *define* $\boldsymbol{u}$ to be the unique solution of

$$-\nu\Delta\boldsymbol{u} + \nabla p_* = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{4.3a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4.3b}$$

where $\boldsymbol{u} \in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega))$ and $p \in L^2(0,T;L^2(\Omega))$ is given by standard elliptic theory for the Stokes equations (see Section 3 above, and Lemma 2.1 in Chapter 1 of [38]).

By the Aubin–Lions compactness lemma (originally due to Aubin [1] and Lions [29]; see also Simon [37], §8, Theorem 5 and Corollary 4), we may extract a subsequence such that $\boldsymbol{B}^m \to \boldsymbol{B}$ strongly in $L^2(0,T;L^2(\Omega))$ and strongly in $C^0([0,T];V^*(\Omega))$. In particular, this gives sense to the initial data with $\lim_{t\to0^+}\boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $V^*(\Omega)$.

We now want to show that $\boldsymbol{u}^m$ does indeed converge to $\boldsymbol{u}$ in the appropriate senses; this will allow us to show that the nonlinear terms involving $\boldsymbol{u}$ converge and thus that the $\boldsymbol{B}$ equation is satisfied in the limit.

Proposition 4.4 *The Galerkin approximations $\boldsymbol{u}^m \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(\Omega))$, and $\boldsymbol{u}^m \rightharpoonup \boldsymbol{u}$ weakly in $L^2(0,T;V(\Omega))$.*

Proof Subtracting the equations for $\boldsymbol{u}^m$ and $\boldsymbol{u}$, we obtain

$$\begin{aligned}-\nu\Delta(\boldsymbol{u}^m-\boldsymbol{u})+\nabla(p_*^m-p_*)&=\nabla\cdot(\boldsymbol{B}^m\otimes\boldsymbol{B}^m-\boldsymbol{B}\otimes\boldsymbol{B})\\&=\nabla\cdot[\boldsymbol{B}^m\otimes(\boldsymbol{B}^m-\boldsymbol{B})+(\boldsymbol{B}^m-\boldsymbol{B})\otimes\boldsymbol{B}].\end{aligned}$$

By elliptic regularity from section 3, we obtain

$$\begin{aligned}\|\boldsymbol{u}^m-\boldsymbol{u}\|_{L^{2,\infty}}&\le c\|\boldsymbol{B}^m\otimes(\boldsymbol{B}^m-\boldsymbol{B})\|_{L^1}+c\|(\boldsymbol{B}^m-\boldsymbol{B})\otimes\boldsymbol{B}\|_{L^1}\\&\le c\|\boldsymbol{B}^m-\boldsymbol{B}\|_{L^2}\left(\|\boldsymbol{B}^m\|_{L^2}+\|\boldsymbol{B}\|_{L^2}\right)\\&\le c(K+M)\|\boldsymbol{B}^m-\boldsymbol{B}\|_{L^2},\end{aligned}$$

where $K=\sup_{m\in\mathbb{N}}\sup_{t\in[0,T]}\|\boldsymbol{B}^m\|_{L^2}$, $M=\sup_{t\in[0,T]}\|\boldsymbol{B}\|_{L^2}$. Squaring and integrating in time yields

$$\int_0^T\|\boldsymbol{u}^m(t)-\boldsymbol{u}(t)\|_{L^{2,\infty}}^2\,\mathrm{d}t\le c\int_0^T\|\boldsymbol{B}^m(t)-\boldsymbol{B}(t)\|_{L^2}^2\,\mathrm{d}t.$$

As the right-hand side converges to zero, so does the left-hand side, and hence $\boldsymbol{u}^m\to\boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(\Omega))$. Let $\boldsymbol{v}$ be the weak limit of $\boldsymbol{u}^m$ in $L^2(0,T;V(\Omega))$; it remains to show that $\boldsymbol{u}=\boldsymbol{v}$. As $V(\Omega)\subset L^2(\Omega)\subset L^{2,\infty}(\Omega)$, we have $(L^{2,\infty})^*(\Omega)\subset L^2(\Omega)\subset V^*(\Omega)$. So if $\boldsymbol{u}^m\rightharpoonup\boldsymbol{v}$ in $L^2(0,T;V(\Omega))$, then $\boldsymbol{u}^m\rightharpoonup\boldsymbol{v}$ in $L^2(0,T;L^{2,\infty}(\Omega))$ (because we are testing with a smaller set of functionals). But $\boldsymbol{u}^m\to\boldsymbol{u}$ strongly (and hence also weakly) in $L^2(0,T;L^{2,\infty}(\Omega))$, and thus by uniqueness of weak limits $\boldsymbol{u}=\boldsymbol{v}$, and the proposition is proved. □

We now proceed to show that the nonlinear terms in the $\boldsymbol{B}$ equation converge. The following proposition is symmetric in $\boldsymbol{B}$ and $\boldsymbol{u}$, and thus applies to both the $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ and $(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}$ terms.

Proposition 4.5 *Suppose that:*

- $\boldsymbol{u}^m\to\boldsymbol{u}$ *and* $\boldsymbol{B}^m\to\boldsymbol{B}$ *(strongly) in* $L^2(0,T;L^{2,\infty}(\Omega))$*; and*
- $\boldsymbol{u}^m$*,* $\boldsymbol{B}^m$ *are uniformly bounded in* $L^\infty(0,T;L^{2,\infty}(\Omega))\cap L^2(0,T;V(\Omega))$.

Then (after passing to a subsequence) $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\overset{*}{\rightharpoonup}(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ *in* $L^2(0,T;V^*(\Omega))$.

Proof For $\phi\in V(\Omega)$,

$$|\langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m],\phi\rangle|\le\|\boldsymbol{u}^m\|_{L^4}\|\boldsymbol{B}^m\|_{L^4}\|\nabla\phi\|_{L^2},$$

so by the weak Ladyzhenskaya inequality (2.2),

$$\|P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\|_{V^*}\le c\|\boldsymbol{u}^m\|_{L^{2,\infty}}^{1/2}\|\nabla\boldsymbol{u}^m\|_{L^2}^{1/2}\|\boldsymbol{B}^m\|_{L^{2,\infty}}^{1/2}\|\nabla\boldsymbol{B}^m\|_{L^2}^{1/2}.$$

Hence $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]$ are uniformly bounded in $L^2(0,T;V^*(\Omega))$. Therefore a subsequence of $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]$ converges weakly-$*$ in $L^2(0,T;V^*(\Omega))$; as usual we relabel this subsequence as the original sequence.

To show that the limit is indeed $(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$, we test with a slightly more regular test function. Let $\phi\in C^0([0,T];V(\Omega))$. Then

$$\begin{aligned}
&\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]-(\boldsymbol{u}\cdot\nabla)\boldsymbol{B},\phi\rangle\,\mathrm{d}t\\
&=\underbrace{\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m-(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}],\phi\rangle\,\mathrm{d}t}_{\mathrm{I}}+\underbrace{\int_0^T \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B},P_m\phi-\phi\rangle\,\mathrm{d}t}_{\mathrm{II}}.
\end{aligned}$$

Clearly II converges since $P_m\phi\to\phi$ in $L^2(0,T;V(\Omega))$. For the first integral, we have

$$\begin{aligned}
\mathrm{I}&=\int_0^T \langle(\boldsymbol{u}^m\cdot\nabla)(\boldsymbol{B}^m-\boldsymbol{B})+((\boldsymbol{u}^m-\boldsymbol{u})\cdot\nabla)\boldsymbol{B},P_m\phi\rangle\,\mathrm{d}t\\
&\le\max_{t\in[0,T]}\|\nabla\phi\|_{L^2}\int_0^T\left(\|\boldsymbol{u}^m\|_{L^4}\|\boldsymbol{B}^m-\boldsymbol{B}\|_{L^4}+\|\boldsymbol{u}^m-\boldsymbol{u}\|_{L^4}\|\boldsymbol{B}\|_{L^4}\right)\mathrm{d}t.
\end{aligned}$$

By the weak Ladyzhenskaya inequality (2.2), and the fact that $\boldsymbol{u}^m\to\boldsymbol{u}$ and $\boldsymbol{B}^m\to\boldsymbol{B}$ in $L^\infty(0,T;L^{2,\infty}(\Omega))$, the right-hand side of the above expression tends to zero. Thus

$$\int_0^T \langle P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]-(\boldsymbol{u}\cdot\nabla)\boldsymbol{B},\phi\rangle\,\mathrm{d}t\to 0\quad\text{for all }\phi\in C^0([0,T];V(\Omega)),$$

and therefore $P_m[(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{B}^m]\overset{*}{\rightharpoonup}(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}$ in $L^2(0,T;V^*(\Omega))$ by uniqueness of weak-$*$ limits. $\square$

Hence $(\boldsymbol{u},\boldsymbol{B})$ is indeed a weak solution of (4.1). Since $\boldsymbol{B}\in L^2(0,T;V(\Omega))$ and $\partial_t\boldsymbol{B}\in L^2(0,T;V^*(\Omega))$, it follows that $\boldsymbol{B}\in C^0([0,T];L^2(\Omega))$ (see [15], §5.9.2, Theorem 3), and hence $\lim_{t\to0^+}\boldsymbol{B}(t)=\boldsymbol{B}_0$ as a limit in $L^2(\Omega)$. This completes the proof of Theorem 4.2 in the case where Ω is a Lipschitz bounded domain in $\mathbb{R}^2$.

4.2 Global existence of weak solutions in $\mathbb{R}^2$

We turn now to the proof of Theorem 4.2 in the case $\Omega=\mathbb{R}^2$. We apply Fourier truncations to the equations, and then show convergence as $R\to\infty$. The arguments are not so different from those in the previous section, so we only outline the main changes.

Define the Fourier truncation $\mathcal{S}_R$ by $\widehat{\mathcal{S}_R f}(\xi)=\mathbb{1}_{B_R}(\xi)\hat{f}(\xi)$, where B_R denotes the ball of radius R centered at the origin. We consider the truncated MHD equations on the whole of $\mathbb{R}^2$ as follows:

$$-\nu\Delta\boldsymbol{u}^R+\nabla p_*^R=(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{B}^R,\tag{4.4a}$$

$$\partial_t\boldsymbol{B}^R+\mathcal{S}_R[(\boldsymbol{u}^R\cdot\nabla)\boldsymbol{B}^R]-\Delta\boldsymbol{B}^R=\mathcal{S}_R[(\boldsymbol{B}^R\cdot\nabla)\boldsymbol{u}^R],\tag{4.4b}$$

$$\nabla\cdot\boldsymbol{u}^R=\nabla\cdot\boldsymbol{B}^R=0,\tag{4.4c}$$

with initial data $\mathcal{S}_R \boldsymbol{B}_0$. Note that, as before, there is no $\mathcal{S}_R$ on the right-hand side of (4.4a). By taking the cutoff initial data as we have, we ensure that, for $t \geq 0$, $\boldsymbol{B}^R$ lies in the space

$$V_R := \{f \in L^2(\mathbb{R}^n) : \hat{f} \text{ is supported in } B_R\},$$

as the truncations are invariant under the flow of the equations; this implies that $\boldsymbol{u}_R \in V_{2R}$. The Fourier truncations act like mollifiers, smoothing the equation; in particular, it is easy to show that

$$F(\boldsymbol{u}^R, \boldsymbol{B}^R) := \mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R]$$

is Lipschitz as a map $F\colon V_{2R} \times V_R \to V_R$. Therefore $\partial_t \boldsymbol{B}^R = G(\boldsymbol{B}^R)$ for some Lipschitz function $G\colon V_R \to V_R$, so by Picard's theorem for infinite-dimensional ODEs, equation (4.4b) will have a unique solution $\boldsymbol{B}^R \in V_R$, and $\boldsymbol{u}^R \in V_{2R}$ is given by equation (4.4a).

Repeating the estimates of Proposition 4.3, with slight modifications to account for the truncations, we again have the following:

$$\begin{aligned}
\boldsymbol{u}^R &\text{ is uniformly bounded in } L^\infty(0,T;L^{2,\infty}(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)),\\
\boldsymbol{B}^R &\text{ is uniformly bounded in } L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)),\\
\partial_t \boldsymbol{B}^R &\text{ is uniformly bounded in } L^2(0,T;V^*(\mathbb{R}^2)).
\end{aligned}$$

Because we are working on $\mathbb{R}^2$, we cannot apply the Aubin–Lions compactness lemma directly (because the embedding $H^1 \subset L^2$ is no longer compact). Instead, there exists a subsequence of $\boldsymbol{B}^R$ that converges strongly in $L^2(0,T;L^2(K))$ for any compact subset $K \subset \mathbb{R}^2$ (see Proposition 2.7 in [8]), and the limit satisfies

$$\boldsymbol{B} \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;V(\mathbb{R}^2)).$$

Thus, we may again define $\boldsymbol{u}$ to be the unique solution of equation (4.3). We now show that $\boldsymbol{u}^R$ converges strongly to $\boldsymbol{u}$ in $L^2(0,T;L^{2,\infty}(K))$ for any compact subset $K \subset \mathbb{R}^2$. This is a little more delicate than the previous case, dealt with in Proposition 4.4: $\boldsymbol{u}$ depends on $\boldsymbol{B}$ on the whole space, but $\boldsymbol{B}^R$ only converges strongly on compact subsets, so we must take care to derive the strong convergence of $\boldsymbol{u}^R$.

Proposition 4.6 *For any compact subset $K \subset \mathbb{R}^2$, $\boldsymbol{u}^R \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{2,\infty}(K))$.*

Proof It suffices to consider $K = B_r$ for any $r > 0$. Set $\mathfrak{B}^R := \boldsymbol{B}^R \otimes \boldsymbol{B}^R$ and $\mathfrak{B} := \boldsymbol{B} \otimes \boldsymbol{B}$. Since $\boldsymbol{B}^R, \boldsymbol{B} \in L^\infty(0,T;L^2(\mathbb{R}^2))$, $\mathfrak{B}^R, \mathfrak{B} \in L^\infty(0,T;L^1(\mathbb{R}^2))$. Moreover, since also $\boldsymbol{B}^R, \boldsymbol{B} \in L^2(0,T;\dot{H}^1(\mathbb{R}^2))$,

$$\|\partial_k(\boldsymbol{B} \otimes \boldsymbol{B})\|_{L^1} = 2\|\boldsymbol{B} \otimes (\partial_k \boldsymbol{B})\|_{L^1} \leq 2\|\boldsymbol{B}\|_{L^2}\|\nabla \boldsymbol{B}\|_{L^2}$$

and since the right-hand side is L^2 in time, $\mathfrak{B}^R, \mathfrak{B} \in L^2(0,T;\dot{W}^{1,1}(\mathbb{R}^2))$. Because $\dot{W}^{1,1}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$, we get

$$\mathfrak{B}^R, \mathfrak{B} \in L^\infty(0,T;L^1(\mathbb{R}^2)) \cap L^2(0,T;L^2(\mathbb{R}^2)).$$

Since $\boldsymbol{B}^R \to \boldsymbol{B}$ strongly in $L^2(0,T;L^2(B_r))$,

$$\begin{aligned}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_r)} &\le \|\boldsymbol{B}^R \otimes (\boldsymbol{B}^R - \boldsymbol{B})\|_{L^1(B_r)} + \|(\boldsymbol{B}^R - \boldsymbol{B}) \otimes \boldsymbol{B}\|_{L^1(B_r)} \\ &\le M\|\boldsymbol{B}^R - \boldsymbol{B}\|_{L^2(B_r)},\end{aligned}$$

where $\|\boldsymbol{B}^R\|_{L^2(\mathbb{R}^2)}, \|\boldsymbol{B}\|_{L^2(\mathbb{R}^2)} \le M$ for all time; hence $\mathfrak{B}^R \to \mathfrak{B}$ strongly in $L^2(0,T;L^1(B_r))$.

Let $G = \partial_k \boldsymbol{U}$ be the derivative of the fundamental solution of the Stokes equation (see Section 3). Then

$$\begin{aligned}\boldsymbol{u}^R(x) - \boldsymbol{u}(x) &= \int_{|y|\le M+r} G(x-y)[\mathfrak{B}^R(y) - \mathfrak{B}(y)]\,\mathrm{d}y \\ &\quad + \int_{|y|>M+r} G(x-y)[\mathfrak{B}^R(y) - \mathfrak{B}(y)]\,\mathrm{d}y \\ &=: I_1(x) + I_2(x).\end{aligned}$$

Now, by Young's inequality,

$$\|I_1\|_{L^{2,\infty}} \le \|G\|_{L^{2,\infty}} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_{M+r})}.$$

If $|x| \le r$, then

$$\begin{aligned}|I_2(x)| &\le \left(\int_{|z|\ge M} |G(z)|^4\right)^{1/4} \|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)} \\ &\le cM^{-1/2}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)},\end{aligned}$$

since $|G(x)| \le c/|x|$. Hence

$$\|\boldsymbol{u}^R - \boldsymbol{u}\|_{L^{2,\infty}(B_r)} \le \|G\|_{L^{2,\infty}}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^1(B_{M+r})} + crM^{-1/2}\|\mathfrak{B}^R - \mathfrak{B}\|_{L^{4/3}(\mathbb{R}^2)}$$

Since $\mathfrak{B}^R - \mathfrak{B}$ is bounded in $L^4(0,T;L^{4/3}(\mathbb{R}^2))$,

$$\int_0^T \|\boldsymbol{u}^R - \boldsymbol{u}\|^2_{L^{2,\infty}(B_r)}\,\mathrm{d}t \le c\int_0^T \|\mathfrak{B}^R - \mathfrak{B}\|^2_{L^1(B_{M+r})}\,\mathrm{d}t + crM^{-1/2}.$$

Thus, given an arbitrary $\delta > 0$ we first pick M sufficiently large so that $crM^{-1/2} < \delta/2$, and then choose R sufficiently large to make the first term at most $\delta/2$. This completes the proof. □

This local strong convergence allows us to pass to the limit in the nonlinear terms: an argument similar to Proposition 4.5 will show that (after passing to a subsequence)

$$\mathcal{S}_R[(\boldsymbol{u}^R \cdot \nabla)\boldsymbol{B}^R] \overset{*}{\rightharpoonup} (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}, \qquad \mathcal{S}_R[(\boldsymbol{B}^R \cdot \nabla)\boldsymbol{u}^R] \overset{*}{\rightharpoonup} (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

in $L^2(0,T;V^*(\mathbb{R}^2))$ (see §2.2.4 of [8] for full details). Thus $(\boldsymbol{u},\boldsymbol{B})$ is indeed a weak solution of (4.1), which completes the proof of Theorem 4.2 in the case $\Omega = \mathbb{R}^2$.

4.3 Uniqueness

We now prove that weak solutions are unique. Note that the following proof applies equally in all three cases of Theorem 1.1.

Proposition 4.7 *Let $(\boldsymbol{u}_j, \boldsymbol{B}_j)$, $j = 1, 2$, be two weak solutions with the same initial condition $\boldsymbol{B}_j(0) = \boldsymbol{B}_0$, such that*

$$\begin{aligned}\boldsymbol{u}_j &\in L^\infty(0,T;L^{2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \boldsymbol{B}_j &\in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\ \partial_t \boldsymbol{B}_j &\in L^2(0,T;V^*(\Omega)).\end{aligned}$$

Then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\boldsymbol{B}_1 = \boldsymbol{B}_2$ as functions in the above spaces.

Proof Take the equations for $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ and subtract: writing $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\boldsymbol{z} = \boldsymbol{B}_1 - \boldsymbol{B}_2$, we obtain

$$0 = \langle \nu\nabla \boldsymbol{w}, \nabla \boldsymbol{v}\rangle + \langle (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{z}\rangle - \langle (\boldsymbol{z} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}_2\rangle, \tag{4.5a}$$

$$\begin{aligned}\langle \partial_t \boldsymbol{z}, \boldsymbol{v}\rangle = \langle \eta\nabla \boldsymbol{z}, \nabla \boldsymbol{v}\rangle &+ \langle (\boldsymbol{B}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}\rangle - \langle (\boldsymbol{z} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}_2\rangle\\ &- \langle (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{v}, \boldsymbol{z}\rangle + \langle (\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{B}_2\rangle.\end{aligned} \tag{4.5b}$$

Let $\boldsymbol{v} = \boldsymbol{w}(t)$ in (4.5a) and use the weak Ladyzhenskaya inequality (2.2) to get

$$\|\boldsymbol{w}\|_{L^4} \le \frac{c}{\nu}\|\boldsymbol{z}\|_{L^2}^{3/4}\|\nabla \boldsymbol{z}\|_{L^2}^{1/4}(\|\nabla \boldsymbol{B}_1\|_{L^2}^{1/4} + \|\nabla \boldsymbol{B}_2\|_{L^2}^{1/4}) \tag{4.6}$$

(using elliptic regularity arguments from Section 3). As $\boldsymbol{z} \in L^2(0,T;V(\Omega))$ and $\partial_t \boldsymbol{z} \in L^2(0,T;V^*(\Omega))$, we can take $\boldsymbol{v} = \boldsymbol{z}(t)$ in (4.5b) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{z}\|_{L^2}^2 + \eta\|\nabla \boldsymbol{z}\|_{L^2}^2\\ &\le c\|\nabla \boldsymbol{z}\|_{L^2}\|\boldsymbol{w}\|_{L^4}(\|\boldsymbol{B}_1\|_{L^4} + \|\boldsymbol{B}_2\|_{L^4}) + c\|\nabla \boldsymbol{z}\|_{L^2}\|\boldsymbol{z}\|_{L^4}(\|\boldsymbol{u}_1\|_{L^4} + \|\boldsymbol{u}_2\|_{L^4}).\end{aligned}$$

Using (4.6) and Young's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{z}\|_{L^2}^2 + \eta\|\nabla \boldsymbol{z}\|_{L^2}^2 \le c(\nu,\eta)\|\boldsymbol{z}\|_{L^2}^2(\|\nabla \boldsymbol{B}_1\|_{L^2}^2 + \|\nabla \boldsymbol{B}_2\|_{L^2}^2). \tag{4.7}$$

As $\boldsymbol{B}_j \in L^2(0,T;V(\Omega))$, and $\boldsymbol{z}_0 = 0$, Gronwall's inequality implies uniqueness of $\boldsymbol{z}$, and hence of $\boldsymbol{w}$ by (4.6). □

This completes the proof of Theorem 4.2.

5 Higher-order regularity estimates

In this section, we prove the second part of Theorem 1.1; that is, that the solution $(\boldsymbol{u}, \boldsymbol{B})$ becomes smooth after an arbitrarily short time $\varepsilon > 0$. In particular, we prove that if we start with initial data in $H^k(\Omega)$, then the solution stays in $H^k(\Omega)$ for all time.

Theorem 5.1 *Let $k \in \mathbb{N}$. Suppose $\boldsymbol{B}_0 \in H^k(\Omega)$ with $\nabla \cdot \boldsymbol{B}_0 = 0$. Then, for any $T > 0$, the unique weak solution of* (4.1) *satisfies*

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0, T; H^k(\Omega)) \cap L^2(0, T; H^{k+1}(\Omega)).$$

Proof We use induction on k; we show only the formal estimates (which can be made rigorous using the same methods as in the last section). First, suppose $\boldsymbol{B}_0 \in H^1(\Omega)$ with $\nabla \cdot \boldsymbol{B}_0 = 0$. Take the inner product of (4.1a) with $-\Delta \boldsymbol{u}$, the inner product of (4.1b) with $-\Delta \boldsymbol{B}$, and add:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{B}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 + \eta\|\Delta \boldsymbol{B}\|_{L^2}^2 \\ &\qquad = \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \Delta \boldsymbol{B}\rangle - \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \Delta \boldsymbol{B}\rangle - \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \Delta \boldsymbol{u}\rangle. \end{aligned}$$

Using Young's inequality we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{B}\|_{L^2}^2 + \frac{\nu}{2}\|\Delta \boldsymbol{u}\|_{L^2}^2 + \frac{\eta}{2}\|\Delta \boldsymbol{B}\|_{L^2}^2 \\ &\leq c\|\nabla \boldsymbol{B}\|_{L^2}^2 \left(\|\boldsymbol{u}\|_{L^{2,\infty}}^2\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{u}\|_{L^2}^2\|\boldsymbol{B}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2\|\nabla \boldsymbol{B}\|_{L^2}^2\right). \end{aligned} \tag{5.1}$$

Since the integral of the last bracket is finite, by Gronwall's inequality we get that $\boldsymbol{B} \in L^\infty(0, T; H^1(\Omega))$. Hence, by (5.1), $\boldsymbol{u}, \boldsymbol{B} \in L^2(0, T; H^2(\Omega))$. Finally, take the inner product of (4.1a) with $\boldsymbol{u}$ to obtain

$$\nu\|\nabla \boldsymbol{u}\|_{L^2} \leq \|\boldsymbol{B}\|_{L^4}^2 \leq c\|\boldsymbol{B}\|_{L^2}\|\nabla \boldsymbol{B}\|_{L^2},$$

and since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0, T; H^1(\Omega))$.

For the induction step, let $k \geq 2$, and let $\boldsymbol{B}_0 \in H^k(\Omega)$ with $\nabla \cdot \boldsymbol{B}_0 = 0$. Suppose

$$\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0, T; H^{k-1}(\Omega)) \cap L^2(0, T; H^k(\Omega)).$$

Take the inner product of (4.1a) with $(-1)^k \Delta^k \boldsymbol{u}$, the inner product of (4.1b) with $(-1)^k \Delta^k \boldsymbol{B}$, and add:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_{H^k}^2 + \nu\|\boldsymbol{u}\|_{H^{k+1}}^2 + \eta\|\boldsymbol{B}\|_{H^{k+1}}^2 \\ &= (-1)^k\left[\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}, \Delta^k \boldsymbol{B}\rangle + \langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \Delta^k \boldsymbol{u}\rangle - \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}, \Delta^k \boldsymbol{B}\rangle\right] \\ &\leq c\left[\|(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\|_{H^k}\|\boldsymbol{B}\|_{H^k} + \|(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\|_{H^k}\|\boldsymbol{u}\|_{H^k} + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\|_{H^k}\|\boldsymbol{B}\|_{H^k}\right] \\ &\leq c\left[\|\boldsymbol{B}\|_{H^k}^2\|\boldsymbol{u}\|_{H^{k+1}} + 2\|\boldsymbol{B}\|_{H^k}\|\boldsymbol{B}\|_{H^{k+1}}\|\boldsymbol{u}\|_{H^k}\right] \\ &\leq \frac{\nu}{2}\|\boldsymbol{u}\|_{H^{k+1}}^2 + \frac{\eta}{2}\|\boldsymbol{B}\|_{H^{k+1}}^2 + c\left(\|\boldsymbol{u}\|_{H^k}^2 + \|\boldsymbol{B}\|_{H^k}^2\right)\|\boldsymbol{B}\|_{H^k}^2, \end{aligned}$$

where we have used the fact that H^k is an algebra for $k \geq 2$. We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|_{H^k}^2 + \nu\|\boldsymbol{u}\|_{H^{k+1}}^2 + \eta\|\boldsymbol{B}\|_{H^{k+1}}^2 \leq c\|\boldsymbol{B}\|_{H^k}^2\left(\|\boldsymbol{u}\|_{H^k}^2 + \|\boldsymbol{B}\|_{H^k}^2\right). \tag{5.2}$$

Since the integral of the last bracket is finite, by Gronwall's inequality we get that $\boldsymbol{B} \in L^\infty(0, T; H^k(\Omega))$, and hence reusing this bound in (5.2) yields $\boldsymbol{u}, \boldsymbol{B} \in L^2(0, T; H^{k+1}(\Omega))$.

If $k \geq 3$ take the inner product of (4.1a) with $(-1)^{k-1}\Delta^{k-1}\boldsymbol{u}$ to obtain

$$\nu\|\boldsymbol{u}\|_{H^k}^2 \leq c\|(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\|_{H^{k-1}}\|\boldsymbol{u}\|_{H^{k-1}} \leq \|\boldsymbol{B}\|_{H^{k-1}}\|\boldsymbol{B}\|_{H^k}\|\boldsymbol{u}\|_{H^{k-1}},$$

since H^{k-1} is an algebra when $k \geq 3$. Since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0,T;H^k(\Omega))$. In the case $k = 2$, since H^1 is not an algebra, we must instead take the inner product of (4.1a) with $-\Delta\boldsymbol{u}$ and estimate as follows:

$$\nu\|\Delta\boldsymbol{u}\|_{L^2}^2 \leq |\langle(\boldsymbol{B}\cdot\nabla)\boldsymbol{B},\Delta\boldsymbol{u}\rangle| \leq \|\boldsymbol{B}\|_{L^4}\|\nabla\boldsymbol{B}\|_{L^4}\|\Delta\boldsymbol{u}\|_{L^2},$$

so

$$\|\Delta\boldsymbol{u}\|_{L^2} \leq \|\boldsymbol{B}\|_{L^2}^{1/2}\|\nabla\boldsymbol{B}\|_{L^2}\|\Delta\boldsymbol{B}\|_{L^2}^{1/2},$$

and since the right-hand side is bounded, $\boldsymbol{u} \in L^\infty(0,T;H^2(\Omega))$. □

An immediate corollary of Theorem 5.1 is that the solution $(\boldsymbol{u},\boldsymbol{B})$ becomes smooth after an arbitrarily short time $\varepsilon > 0$, which completes the proof of Theorem 1.1.

Corollary 5.2 *Given $T > \varepsilon > 0$ and $k \in \mathbb{N}$, the unique weak solution of* (4.1) *satisfies $\boldsymbol{u},\boldsymbol{B} \in L^\infty(\varepsilon,T;H^k(\Omega))$.*

Proof Fix $\varepsilon > 0$. We already know that $\boldsymbol{u},\boldsymbol{B} \in L^2(0,T;H^1(\Omega))$, so for some time $t_1 < \varepsilon/2$, $\boldsymbol{u}(t_1),\boldsymbol{B}(t_1) \in H^1(\Omega)$. Applying Theorem 5.1, we obtain

$$\boldsymbol{u},\boldsymbol{B} \in L^\infty(\varepsilon/2,T;H^1(\Omega)) \cap L^2(\varepsilon/2,T;H^2(\Omega)).$$

Furthermore, if we know that

$$\boldsymbol{u},\boldsymbol{B} \in L^\infty(\varepsilon(1-2^{1-k}),T;H^{k-1}(\Omega)) \cap L^2(\varepsilon(1-2^{1-k}),T;H^k(\Omega)),$$

then there is some time t_k such that $\varepsilon(1-2^{1-k}) < t_k < \varepsilon(1-2^{-k})$ and $\boldsymbol{u}(t_k),\boldsymbol{B}(t_k) \in H^k(\Omega)$, and so applying Theorem 5.1, we obtain

$$\boldsymbol{u},\boldsymbol{B} \in L^\infty(\varepsilon(1-2^{-k}),T;H^k(\Omega)) \cap L^2(\varepsilon(1-2^{-k}),T;H^{k+1}(\Omega)).$$

The result follows by induction on k. □

6 The 3D case

It is straightforward to adapt the methods of Section 4 to the 3D case to prove global existence — but *not* uniqueness — of at least one weak solution to (1.1) in 3D. Indeed, for $\Omega \subset \mathbb{R}^3$ in the analogue of Theorem 1.1, given an initial condition $\boldsymbol{B}_0 \in H(\Omega)$ there exists at least one weak solution $(\boldsymbol{u},\boldsymbol{B})$ of (1.1) on $(0,T)$ with

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(0,T;L^{3/2,\infty}(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\boldsymbol{B} &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega)),\\
\partial_t\boldsymbol{B} &\in L^{24/19}(0,T;V^*(\Omega)),
\end{aligned}$$

that satisfies the initial data in the sense that $\lim_{t\to 0^+}\boldsymbol{B}(t) = \boldsymbol{B}_0$ as a limit in $V^*(\Omega)$.

The key differences come from the elliptic regularity and the consequent interpolation inequalities. In 3D, the solution of the Stokes equation (3.1) satisfies $\boldsymbol{u} \in L^{3/2,\infty}(\Omega)$ whenever $\boldsymbol{f} \in L^1(\Omega)$. The standard 3D Ladyzhenskaya inequality

$$\|f\|_{L^4} \le c\|f\|_{L^2}^{1/4}\|f\|_{H^1}^{3/4} \tag{6.1}$$

is then sufficient; using that and the interpolation inequality

$$\|f\|_{L^4} \le c\|f\|_{L^{3/2,\infty}}^{1/6}\|f\|_{L^6}^{5/6} \le c\|f\|_{L^{3/2,\infty}}^{1/6}\|f\|_{H^1}^{5/6} \tag{6.2}$$

it is straightforward to show the corresponding energy estimates to Proposition 4.3 in the case when Ω is a bounded domain in $\mathbb{R}^3$: $\boldsymbol{u}^m$, $\boldsymbol{B}^m$ and $\partial_t \boldsymbol{B}^m$ are uniformly bounded in the corresponding spaces as above.

The Aubin-Lions compactness lemma then shows that $\boldsymbol{B}^m \to \boldsymbol{B}$ strongly in both $L^2(0,T;L^2(\Omega))$ and $C^0([0,T];V^*(\Omega))$ (so that the initial data is attained in this sense). It is simple to adjust Proposition 4.4 to show that $\boldsymbol{u}^m \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{3/2,\infty}(\Omega))$, using (6.1) and (6.2). Modifying Proposition 4.5 yields convergence of the nonlinear terms in $L^{24/19}(0,T;V^*(\Omega))$, and hence $(\boldsymbol{u},\boldsymbol{B})$ is a weak solution of equations (1.1).

It is also routine to modify the method of Section 4.2 to prove existence in the case $\Omega = \mathbb{R}^3$. One modifies Proposition 4.6 to show that $\boldsymbol{u}^R \to \boldsymbol{u}$ strongly in $L^2(0,T;L^{3/2,\infty}(K))$ for any compact subset $K \subset \mathbb{R}^3$, by using the embedding $\dot{W}^{1,1}(\mathbb{R}^3) \subset L^{3/2}(\mathbb{R}^3)$, and $|G| = |\partial_k \boldsymbol{U}| \le c/|x|^2 \in L^{3/2,\infty}(\mathbb{R}^3)$.

7 Non-resistive case ($\boldsymbol{\eta = 0}$)

In the above we have developed an essentially complete theory of existence, uniqueness, and regularity for the system (1.1) when $\eta > 0$.

The non-resistive case ($\eta = 0$) is much more difficult, and analogous to the vorticity formulation of the 3D Euler equations in the same way that the resistive system has similarlities with the 3D Navier-Stokes system (as discussed in the introduction). Two-dimensional models with similar structure to these canonical 3D equations (such as the 2D SQG equation [11]) have attracted considerable attention in recent years, and we will present an analysis of (1.1) with $\eta = 0$ in our future paper [17], in which we prove local existence of solutions to (1.1) with $\eta = 0$ in $H^s(\mathbb{R}^n)$ for $s > n/2$.

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